C2 Exercise 7E (the sum to infinity of a geometric series)

Note Title 28/04/2013

1 Find the sum to infinity, if it exists, of the following series:

a $1 + 0.1 + 0.01 + 0.001 + \ldots$

$a = 1 \quad r = \frac{1}{10}$ $\quad S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = \frac{10}{9} = 1.\dot{1}$

b $1 + 2 + 4 + 8 + 16 + \ldots$

$a = 1 \quad r = 2$ $\quad S_\infty$ does not exist.

c $10 - 5 + 2.5 - 1.25 + \ldots$

$a = 10 \quad r = -\frac{1}{2}$

$S_\infty = \frac{a}{1-r} = \frac{10}{1 - -\frac{1}{2}} = \frac{20}{3} = 6.\dot{6}$

d $2 + 6 + 10 + 14 + \ldots$

$a = 2 \quad d = 4$ (it's an arithmetic sequence $u_n = 2 + 4(n-1)$)

S_∞ doesn't exist.

e $1 + 1 + 1 + 1 + 1 + \ldots$

$a = 1 \quad r = 1$

S_∞ doesn't exist $\quad |r|$ must be strictly less than 1.

f $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

$a = 3 \quad r = \frac{1}{3}$

$S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{1}{3}} = \frac{9}{2} = 4.5$

1g $0.4 + 0.8 + 1.2 + 1.6 + \ldots$

$a = 0.4 \quad d = 0.4$ it's an A.P. again with $u_n = 0.4n$

S_∞ doesn't exist.

h $9 + 8.1 + 7.29 + 6.561 + \ldots$

$a = 9 \quad r = 0.9$

$$S_\infty = \frac{a}{1-r} = \frac{9}{1-0.9} = 90.$$

i $1 + r + r^2 + r^3 + \ldots$

$a = 1 \quad r = 'r'$ $\quad S_\infty = \frac{1}{1-r}$ provided that $|r| < 1$

j $1 - 2x + 4x^2 - 8x^3 + \ldots$

$a = 1 \quad r = -2x$ $\quad S_\infty = \frac{1}{1+2x}$ provided $|-2x| < 1$

which means

$|x| < \frac{1}{2}$

2 Find the common ratio of a geometric series with a first term of 10 and a sum to infinity of 30.

$a = 10 \quad r$ is unknown $\quad S_\infty = \frac{a}{1-r} = 30$

$$\frac{10}{1-r} = 30 \Rightarrow \frac{10}{30} = 1 - r$$

$$\Rightarrow 1 - r = \frac{1}{3}$$

$$\Rightarrow r = \frac{2}{3}.$$

3 Find the common ratio of a geometric series with a first term -5 and a sum to infinity of -3.

$a = -5$ $\quad$ r is unknown, to be found $\quad$ $S_\infty = -3$

$$S_\infty = \frac{a}{1-r} \quad \Rightarrow \quad -3 = \frac{-5}{1-r}$$

$$\Rightarrow \quad 1 - r = \frac{-5}{-3}$$

$$\Rightarrow \quad 1 - \frac{5}{3} = r$$

$$\Rightarrow \quad r = -2/3.$$

4 Find the first term of a geometric series with a common ratio of $2/3$ and a sum to infinity of 60.

a is unknown, to be found, $r = \frac{2}{3}$, $S_\infty = 60$

$$S_\infty = \frac{a}{1-r} \quad \Rightarrow \quad 60 = \frac{a}{1 - 2/3}$$

$$\Rightarrow \quad 60 = \frac{a}{1/3}$$

$$\Rightarrow \quad a = \frac{1}{3} \times 60$$

$$\Rightarrow \quad a = 20$$

5 Find the first term of a geometric series with a common ratio of $-\frac{1}{3}$ and a sum to infinity of 10.

a is unknown, to be found ; $r = -\frac{1}{3}$; $S_\infty = 10$.

$$S_\infty = \frac{a}{1-r} \Rightarrow 10 = \frac{a}{1 - -\frac{1}{3}}$$

$$\Rightarrow 10 = \frac{a}{\frac{4}{3}}$$

$$\Rightarrow \frac{4}{3} \times 10 = a$$

$$\Rightarrow a = \frac{40}{3} = 13\tfrac{1}{3}.$$

6 Find the fraction equal to the recurring decimal 0.232323232...

Series method:

$$0.232323232\ldots = \frac{23}{100} + \frac{23}{10000} + \frac{23}{1000000} + \ldots$$

so it forms a series with $a = \frac{23}{100}$ $r = \frac{1}{100}$

$$S_\infty = \frac{a}{1-r} \Rightarrow 0.\dot{2}\dot{3} = \frac{\frac{23}{100}}{1 - \frac{1}{100}}$$

$$= \frac{23}{100} \div \frac{99}{100}$$

$$= \frac{23}{100} \times \frac{100}{99}$$

$$\Rightarrow 0.\dot{2}\dot{3} = \frac{23}{99}.$$

There's a different algebraic method taught at GCSE:

put $x = 0.\dot{2}\dot{3}$

$\Rightarrow 100x = 23.\dot{2}\dot{3}$ This is far neater.

$\Rightarrow 99x = 23$ $\Rightarrow x = \frac{23}{99}.$

7 Find $\sum_{r=1}^{\infty} 4(0.5)^r$

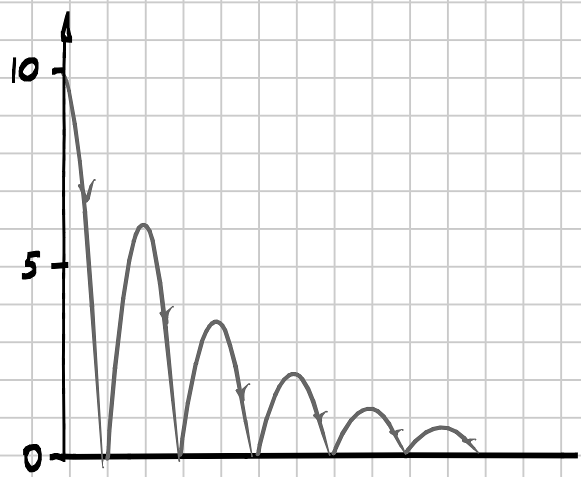
$a = 4 \times 0.5 = 2$

$r = 0.5$

$$S_\infty = \frac{a}{1-r} \Rightarrow \sum_{r=1}^{\infty} 4(0.5)^r = \frac{2}{1-0.5} = 4.$$

8 A ball is dropped from a height of 10m. It bounces to a height of 6m, then 3.6m and so on following a geometric sequence.

Find the total distance travelled by the ball.



$a = 10$

$r = \frac{6}{10} = 0.6$

$$S_\infty = \frac{10}{1-0.6} = \frac{10}{0.4} = \frac{100}{4} = 25$$

The ball travels $2S_\infty - 10$m since each bounce contributes the distance on the way up <u>and</u> on the way down, except the first 10m which is only downward.

Total distance then is $2 \times 25 - 10 = 40$m.

9 The sum to three terms of a geomeric series is 9 and its sum to infinity is 8.

What can you deduce about the common ratio? Why?

The common ratio must be negative ($r<0$) since adding more terms reduces the sum. ($S_\infty < S_3$)

Find the first term and the common ratio.

$S_3 = 9 \qquad S_\infty = 8 \qquad a = \text{unknown} \qquad r = \text{unknown}$

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow 9 = \frac{a(1-r^3)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \Rightarrow 8 = \frac{a}{1-r}$$

$$\Rightarrow 9 = 8(1-r^3)$$

$$\Rightarrow 9 = 8 - 8r^3$$

$$\Rightarrow 8r^3 = -1$$

$$\Rightarrow r^3 = -1/8$$

$$\Rightarrow r = -1/2.$$

Return to $S_\infty = \frac{a}{1-r} \Rightarrow 8 = \frac{a}{1-(-1/2)}$

$$\Rightarrow a = \frac{3}{2} \times 8 = 12$$

Hence first term is 12 and common ratio $-\frac{1}{2}$.

10 The sum to infinity of a geometric series is three times the sum to 2 terms. Find all possible values of the common ratio.

a = unknown , r = unknown , put $S_2 = x \Rightarrow S_\infty = 3x$

$$S_2 = \frac{a(1-r^2)}{1-r} = x \qquad S_\infty = \frac{a}{1-r} = 3x$$

$$\Rightarrow \quad 3x(1-r^2) = x$$

$$\Rightarrow \quad 1-r^2 = \frac{1}{3}$$

$$\Rightarrow \quad r^2 = \frac{2}{3}$$

$$\Rightarrow \quad r = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$$